Chapter 3: Limits and Continuity

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1 Limits

2 Continuity
Informal definition of Limits

The limit of \( f(x) \), as \( x \) approaches \( c \), is equal to \( L \) means that \( f(x) \) can be made arbitrarily close to \( L \) whenever \( x \) is sufficiently close to \( c \). We denote this by

\[
\lim_{{x \to c}} f(x) = L.
\]

If \( L \) is a finite number, we say that the limit exists and that \( f(x) \) converges to \( L \) as \( x \) tends to \( c \).

Some Examples:

\[
\lim_{{x \to 1}} (x^2 + 1), \quad \lim_{{x \to 2}} \frac{x^2 - 4}{x - 2}.
\]

Note that we choose \( x \) is very close to \( x \) but not equal to \( c \). We do not simply plug \( c \) into \( f(x) \).
One sided limits

\[
\lim_{x \to c^+} f(x) = L \text{ when } x \text{ approaches } c \text{ from the right.}
\]

\[
\lim_{x \to c^-} f(x) = L \text{ when } x \text{ approaches } c \text{ from the left.}
\]

Examples: Find \(\lim_{x \to 0^+} \frac{|x|}{x}\), \(\lim_{x \to 0^-} \frac{|x|}{x}\)
Example: consider \( \lim_{x \to 0} \frac{1}{x^2} \).
Example: consider \( \lim_{x \to 0} \frac{1}{x^2} \).

**Remark**

\[
\lim_{x \to c} f(x) = +\infty \text{ if } f(x) \text{ increases without bound as } x \to c.
\]

\[
\lim_{x \to c} f(x) = -\infty \text{ if } f(x) \text{ decreases without bound as } x \to c.
\]

Examples:

\[
\lim_{x \to 2^+} \frac{1}{x - 2} = +\infty,
\]

\[
\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty.
\]
Some Examples:

\[
\lim_{x \to \infty} \frac{x}{x + 1},
\]

\[
\lim_{x \to \infty} \frac{2x^2 - x + 1}{x^2 - x + 1},
\]

\[
\lim_{x \to +\infty} \sqrt{x^2 + 3x - 1} - \sqrt{x^2 + 1}.
\]
Limit Laws

See the page 124
The sandwich theorem

**Theorem**

If \( f(x) \leq g(x) \leq h(x) \) for \( x \) close to \( c \) and if

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,
\]

then

\[
\lim_{x \to c} g(x) = L.
\]

Example: show that

\[
\lim_{x \to 0} x \sin \frac{1}{x} = 0.
\]
A fundamental trigonometric limit

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]

Example: show that

\[ \lim_{x \to 0} \frac{1 - \cos x}{x} = 1. \]
Problems

3.1.3 Problems page 127

3.3.1 Problems page 142

3.4.1 Problems page 148
Problems

3.1.3 Problems page 127

3.3.1 Problems page 142

3.4.1 Problems page 148

Home work
Exs 9, 18, 26, 31, 34, 48 page 127

Exs 5, 11, 15, 27 page 142

Exs 4, 13 page 148
Definition

A function $f$ is said to be continuous at $c$ if

$$\lim_{{x \to c}} f(x) = f(c).$$

If not, the function $f$ is discontinuous at $c$. Some examples can be found on page 131.
A function \( f \) is said to be continuous at \( c \) if

\[
\lim_{x \to c} f(x) = f(c).
\]

That means,

- \( f \) is well defined at \( c \);
- \( \lim_{x \to c} f(x) \) exists;
- \( \lim_{x \to c} f(x) = f(c) \).

If not, the function \( f \) is discontinuous at \( c \).

Some examples
Definition

A function $f$ is said to be continuous at $c$ if

$$\lim_{{x \to c}} f(x) = f(c).$$

That means,

- $f$ is well defined at $c$;
- $\lim_{{x \to c}} f(x)$ exists;
- $\lim_{{x \to c}} f(x) = f(c)$.

If not, the function $f$ is discontinuous at $c$.

Some examples
Read yourself Def. page 131.
The Intermediate Value Theorem (I.V.T.)

Let $f$ be a continuous function on the closed interval $[a, b]$. For $L$ is any real number between $f(a)$ and $f(b)$, then there exists at least one number $c \in (a, b)$ such that $f(c) = L$.

Example: let

$$f(x) = 3 + \sin x \text{ for } 0 \leq x \leq \frac{3\pi}{2}.$$ 

Show that there exists at least one point $c$ in $(0, \frac{3\pi}{2})$ such that $f(x) = \frac{5}{2}$. 
3.5.2. A final Remark on continuous function;
The Bisection method: example 2 page 150.
Problems

Examples 2, 3, 4, 5 page 130-132: remove discontinuities (discontinuity removed?)

3.2.3 Problems page 137

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Problems

Examples 2, 3, 4, 5 page 130-132: remove discontinuities (discontinuity removed?)

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Home work
Exs 8, 11, 16, 22, 27, 31, 37 page 137

Exs 3, 4 page 152
THANK YOU FOR YOUR ATTENTION!